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ENUMERATION OF OPTIMALLY LABELLED n-POSETS OF LINEAR DISCREPANCY TWO

GAB-BYUNG CHAE *

Abstract. Any labelled n-poset of linear discrepancy two can be written in terms of matrix representations. But there are matrices which are not the adjacency matrices of the comparability graphs of any poset at all, say *non-poset matrices* of order n, and matrices which are of tightness two but of linear discrepancy one, say *non-optimal labelled matrices* of order n in matrix representations. In this paper, the number of optimally labelled n-posets of linear discrepancy two is obtained by counting the number of non-poset matrices and the number of non-optimal labelled matrices.

1. Introduction

We denote by $P = (X, \leq_P)$ a partially ordered set(simply say a poset) P with a given ground set X and an order relation $\leq_P$. For $x, y \in X$, if $(x, y) \in \leq_P$, we say that x is *comparable* with y, written as $x \leq_P y$. If $(x, y) \notin \leq_P$, we say that x is *incomparable* to y, written as $x \parallel_P y$. If $|X| = n$ for a positive integer n, then P is called an n-poset. The chain of order n, written as $\mathbf{n} = (X, \leq_{\mathbf{n}})$, is a poset which is $|X| = n$ and $x \leq_{\mathbf{n}} y$ or $y \leq_{\mathbf{n}} x$ for all $x, y \in X$. For a given poset $P = (X, \leq_P)$, a function $f : X \longrightarrow \mathbb{Z}$ is called an *isotone* if it preserves the order-relation of P, i.e. $f(x) \leq f(y)$ if $x \leq_P y$ for $x, y \in X$. If the image of a bijective isotone f of an n-poset P is $[n] = \{1, ..., n\}$, we call this f a *natural labelling (or labelling)* of P, $f(x)$ *the label of* $x \in X$, and a pair (P, f) a *naturally labelled poset*. A poset $E = (X, \leq_E)$ is called an extension of $P = (X, \leq_P)$ if $\leq_P \subseteq \leq_E$. If an extension of $P = (X, \leq_P)$ with $|X| = n$ is a chain, it is called a linear extension of P of order n. Thus we may consider a labelling f of $P = (X, \leq_P)$ as a linear extension of P.

The *tightness* of a labelling f on a poset $P = (X, \leq_P)$, when written as $T_f(P)$, is the maximum difference between the labels of incomparable pairs of P, i.e., $T_f(P) = \max_{x||y} |f(x) - f(y)|$. We define $T_f(\mathbf{n}) = 0$ for a chain $\mathbf{n}$. The *linear discrepancy* (briefly say 'LD') of $P = (X, \leq_P)$, when written as $ld(P)$, is the minimum tightness over all labellings on P, i.e.,

$$ld(P) = \min_{f \in \mathcal{F}} T_f(P) = \min_{f \in \mathcal{F}} \max_{x\|_P y} |f(x) - f(y)|,$$

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where $\mathcal{F}$ is the set of all natural labellings of P. A labelling $f \in \mathcal{F}$ with $ld(P) = T_f(P)$ is called an *ld-optimal labelling*(or *optimal labelling*) of P, and a pair (P, f) is called an *optimally labelled poset*.

There is a matrix representation of a poset with a natural labelling and its linear discrepancy, which can be referred to as J. Negger and H. S. Kim[8] and S.-M. Kim, G.-B. Chae, and M. Cheong[7] for detailed explanation. If $P = (X, \leq_P)$ be a naturally labelled poset with $|X| = n$ and a given natural labelling f, then there is an $n \times n$ $(0,1)$-matrix $U_P^f = [a_{i,j}]$, called a (*naturally*) *labelled matrix* of P with f, defined as follows.

$$(1) \qquad a_{i,j} = \begin{cases} 1, & \text{if } f^{-1}(i) \leq f^{-1}(j) \text{ for } i,j \in [n]; \\ 0, & \text{otherwise.} \end{cases}$$

and vice versa. In point of view of graph theory, the labelled matrix of a labelled n-poset is the adjacency matrix of the comparability graph of a labelled n-poset (or the adjacency matrix of the complement of the incomparability graph of a labelled n-poset). Hence, in this paper, we identify (P, f) with U_P^f for a poset P with a labelling f. If $T(U_P^f)$ denotes $\max\{j - i : a_{i,j} = 0,\ j \geq i\}$ then $T_f(P) = T(U_P^f)$, since $T_f(P) = \max\{|f(x) - f(y)| : x, y \in X,\ \ x \parallel_P y\}$. Hence we have

$$ld(P) = \min_f \{T(U_P^f)\}$$

If f is an optimal labelling on P, then we have $ld(P) = T(U_P^f)$; hence, we may call such an U_P^f an *optimally labelled matrix* of P.

The problem of characterization of posets with respect to the linear discrepancy was firstly suggested by P. Fishburn, P. Tanenbaum, and A. Trenk[3] together with the definition of linear discrepancy. They characterized the posets of linear discrepancy 1. Also, they showed that the linear discrepancy of any poset P is equal to the bandwidth of the incomparability graph of P. Although there are researches on the problem of characterization of posets with respect to the linear discrepancy, there has been no enumeration result of posets with respect to the linear discrepancy. In this paper, the number of optimally labelled n-posets of LD two is obtained by counting optimally labelled matrices U_P^f of LD two. Consider a labelled matrix of LD less than or equal to two of labelled n-poset as in Figure 1. If the circle-numbered places in Figure 1 are all filled up with 0's and all other places are filled up with 1's, then the matrix of order n is called the labelled $(n, 2)$-*simple matrix* since the matrix is the adjacency matrix of the simplest comparability graph of a labelled n-poset of LD two(defined in [6]) in the sense of the following assertion. It is easy to notice that each of the labelled matrices U_P^f of LD less than or equal to two is an extension of the labelled $(n, 2)$-simple matrix [6]. There are $(n-1) + (n-2) = 2n - 3$ available entries on the first two diagonal lines(say first diagonal line and the second diagonal line of the matrix) above the main diagonal line for the extension of labelled $(n, 2)$-simple matrix. The labelled matrices U_P^f of LD less than or equal to two can be obtained from $(n, 2)$-simple matrix by adding some 0's or 1's on these $2n - 3$ entries. We call each of these $2n - 3$ entries an *asset position* of

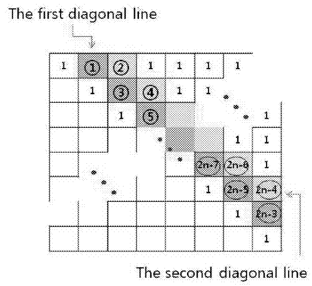

FIGURE 1. Asset positions for a matrix

U_P^f. Hence the number of the labelled matrices U_P^f of LD less than or equal to two is 2^{2n-3}.

Among the labelled matrices of LD less than or equal to two, there are 5 different types of labelled matrices to be examined as follows. First, among the 2^{2n-3} labelled matrices, there are labelled matrices which represent optimally labelled posets of LD one. The number of these matrices of order n is denoted by $LD_1(n)$ which is counted by Theorem 1.6. Second, the number of labelled matrices of posets of order n with a given LD is less than the number of the adjacency matrices of the complement of all the graphs with the same value of bandwidth. The reason is that there are matrices which are not the adjacency matrices of the comparability graphs of any poset at all but those are the complement of adjacency matrices of graphs of the same value of bandwidth. These are called *non-poset matrices* of order n . The number of these matrices of order n is denoted by $NP(n)$. Third, there are labelled matrices which represent the labelled posets of tightness two but of LD one, called *non-optimal labelled matrices.* The number of these labelled matrices of order n is denoted by $NOL(n)$. Fourth, we have labelled matrices with an optimal labelling which represent posets of LD two. The number of these labelled matrices of order n is denoted by $LD_2(n)$, which is the number we want to find. Fifth, a labelled matrix with all the entries on the upper triangle are 1s and all the others are 0s, represents a chain of LD zero.

The following lemma can be used for determining whether a given upper triangular (0,1)-matrix satisfies the transitivity for being a poset.

Lemma 1.1. [7] *Let $U = [a_{i,j}]$ be an $n \times n$ upper triangular $(0,1)$-matrix whose diagonal entries are all* 1*s. Let*

$$r_i = (a_{i,i+1}, a_{i,i+2}, \cdots, a_{i,k}, \cdots, a_{i,j-1}),$$

and

$$c_j = (a_{i+1,j}, a_{i+2,j}, \cdots, a_{k,j}, \cdots, a_{j-1,j})$$

Then U uniquely represents a naturally labelled n-poset if and only if U satisfies the property that $r_i \cdot c_j = 0$ if $a_{i,j} = 0$ for $i, j \in [n]$ with $i < j$.

The following lemma is for a characterization of non-poset matrices.

Lemma 1.2. *A non-poset matrix of order n and of LD less than or equal to two is the matrix which has* $\begin{matrix} 1 & 0 \\ & 1 \end{matrix}$ *in the asset positions consecutively with placing* 0 *on the second diagonal line.*

Proof. If a matrix has $\begin{matrix} 1 & 0 \\ & 1 \end{matrix}$ then it is not a labelled matrix of any poset at all since it fails to satisfy the transitivity law for being a poset by Lemma 1.1. □

Now the non-optimal labelled matrices are needed to be characterized, and hence we need the following definition in [2].

Definition 1.3. *Let P be a poset with $ld(P) = t$. If $ld(P \setminus \{x\}) \leq t-1$ for any $x \in P$, P is called t-**irreducible**.*

In [4] and [5], the 3-irreducible posets of width 2 and 3 were characterized, respectively, and hence posets of LD 2 are characterized as follows.

Theorem 1.4. *P is a poset with $ld(P) = 2$ if and only if P has a 2-irreducible poset as a subposet but no 3-irreducible poset as a subposet.*

Normally, this characterization is essential to count the number of posets of LD 2, but it is not in this paper. Since we use optimally labelled matrices as in Figure 1, the posets of tightness three and LD greater than or equal to three are exempted from counting. Hence to determine whether a poset P(or corresponding labelled matrix) is LD two or not, it is sufficient to find out if it contains one of $\mathbf{1+1+1}$, $\mathbf{1+3}$, or $\mathbf{2+2}$ as a subposet, stated as in the following proposition [2].

Proposition 1.5. *U_P^f is a labelled matrix of order n and of LD greater than or equal two if and only if P has one of $\{\mathbf{1+1+1}, \mathbf{1+3}, \mathbf{2+2}\}$.*

The number of optimally labelled n-posets of LD one is as follows(seen in [7]) since each of n-posets of LD one has the unique optimal labelling and there are $n-1$ positions in the upper diagonal entries where 0 is allowed.

Theorem 1.6. [7] *The number of n-posets of LD one is $2^{n-1} - 1$.*

Since the total number of all matrices in five cases above is 2^{2n-3}, we have the following lemma.

Lemma 1.7. *For a positive integer n, the number of the optimally labelled n-posets of LD two, $LD_2(n)$, is $2^{2n-3} - LD_1(n) - NP(n) - NOL(n) - 1$.*

2. The number of non-poset matrices

The number $NP(n)$ is obtained inductively on a positive integer n. Let us say $N_n = NP(n)$ in this section for convenience. A Non-poset matrix of order n may be obtained by appending the n-th column and the n-th row to a non-poset matrix of order $(n-1)$ at a time. Let $U = [a_{i,j}]$ be a non-poset matrix of order n.

Case 1 : The non-poset matrices of order n which are obtained from those of order $n-1$ or less are counted. Since the $a_{n-2,n}$ and $a_{n-1,n}$ of U could have 0 or 1, we have four cases to consider. However if the matrices of order $n-1$ or less are already non-poset matrices, then we already have non-poset matrices of order n regardless of the values of $a_{n-2,n}$ and $a_{n-1,n}$, which is counted by $4N_{n-1}$(See Case 1 in Figure 2).

Case 2 : Now the following cases are related to the purely newly generated non-poset matrix of order n not caused by the non-poset matrices of order $(n-1)$ or less but caused by adding new n-th row and column. Mind that we may have $a_{n-2,n} = 0$ or 1, and $a_{n-1,n} = 0$ or 1 so that we have four cases to consider. It is quite clear that the three cases except $a_{n-2,n} = 0$ and $a_{n-1,n} = 1$ do not make any contribution to the number of non-poset matrices of order n.

Case 2-1 : We count the number of non-poset matrices of order n which are generated by the case when $a_{n-3,n-1} = 1$, $a_{n-2,n-1} = 1$, $a_{n-2,n} = 0$, and $a_{n-1,n} = 1$. The other components in the first and second diagonal lines are denoted by $\times$ in Case 2-1 of Figure 2. The number of those components is $2n-7$ so that it generates 2^{2n-7} matrices which contribute to the number of non-poset matrices of order n. But in this number, we need to subtract N_{n-2} which is already counted. Therefore, the number is $2^{2n-7} - N_{n-2}$.

Case 2-2: The non-poset matrices of order n which are generated by the case when $a_{n-3,n-1} = 0$, $a_{n-2,n-1} = 1$, $a_{n-2,n} = 0$, and $a_{n-1,n} = 1$ are considered. Note that the component $a_{n-3,n-2}$ should be 0; otherwise, this case is already counted by N_{n-1}. Then the component $a_{n-4,n-2}$ has two options to be 0 or 1. Other components in the first or the second diagonal lines are denoted by $\times$ in Case 2-2 of Figure 2. The number of those components is $2n-8$ which contributes 2^{2n-8} to the number of non-poset matrices of order n. From this number we need to subtract $2 \cdot N_{n-3}$ which is already counted. Therefore, the number is $2(2^{2n-9} - N_{n-3})$.

Case 2-3 : When $a_{n-3,n-1} = 0$, $a_{n-2,n-1} = 0$, $a_{n-2,n} = 0$, and $a_{n-1,n} = 1$, there is no contribution to the number of non-poset matrices of order n.

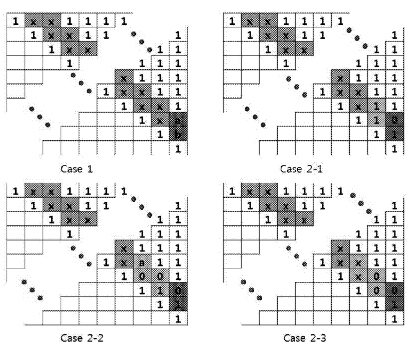

FIGURE 2. Construct non-poset matrices

In all, we have the recurrence relation for non-poset matrices, as follows.

Theorem 2.1. *Let N_n be the number of non-poset matrices of order n. Then*

$$N_n = 4N_{n-1} - N_{n-2} - 2N_{n-3} + 3 \cdot 4^{n-4}, \; n \geq 4. \tag{2}$$

Equivalently,

$$\begin{aligned} A(x) &= \frac{\frac{3}{4^4(1-4x)} - 3/4^4 - 3x/4^3 - 3x^2/16 + x^3/4}{1 - 4x + x^2 + 2x^3} \\ &= \frac{1}{8} - \frac{1-x}{4(1-3x-2x^2)} + \frac{1}{8(1-4x)} \end{aligned} \tag{3}$$

where $A(x) = \sum N_n x^n$, $N_0 = 0$, $N_1 = 0$, $N_2 = 0$, and $N_3 = 1$.

By solving (2) for N_n, we have the following theorem.

Theorem 2.2. *For a positive integer n,*

(4)

$$NP(n) = N_n = 2^{2n-3} + \frac{7\sqrt{17}+17}{1088}\left(\frac{3-\sqrt{17}}{2}\right)^{n+1} + \frac{17-7\sqrt{17}}{1088}\left(\frac{3+\sqrt{17}}{2}\right)^{n+1},$$

which leads to $N_n \sim 2^{2n-3}$ when n tends to infinity.

The partial fraction method is used to find approximation for the number N_n. From Equation (3), we have

$$\begin{aligned} A(x) &= \frac{1}{8} - \frac{0.0436093...}{0.280776... - x} \\ &\quad - \frac{0.168609...}{1.78078... + x} - \frac{1}{8}\frac{1}{4x-1}. \end{aligned} \tag{5}$$

Then the coefficient of x^n in $\frac{1}{8}\frac{1}{4x-1}$ is

$$2^{2n-3}$$

and the coefficient of x^n in $\frac{0.168609...}{1.78078...+x}$ tends to zero as n tends to ∞. The coefficient of x^n in $-\frac{0.0436093...}{0.280776...-x}$ is $-0.0436093 \cdot 0.280776^{-1-n}$. Hence we have the following result.

Theorem 2.3. *The asymptotic number of non-poset matrices is*

$$N_n \sim -C_1 \cdot r_1^{-1-n} + 2^{2n-3} \tag{6}$$

where $C_1 = 0.0436093...$ and $r_1 = 0.280776...$

3. The number of non-optimal labelled matrices

Note that a non-optimal labelled matrix has one 0 at an asset position in the second diagonal line, since it represents the poset of tightness two. Again we begin with the labelled $(n, 2)$-simple poset matrix of LD two in Figure 1. Suppose that U_P^f is a labelled matrix with a labelling f of a poset P of tight two. The possible cases to be considered are illustrated in Figure 3. If U_P^f has 101 in which the 0 is located at an asset position in the second diagonal line, then it is a non-poset matrix(see Figure 3(a)), which has already been counted in Section 2. If U_P^f has 001 or 100 where one 0 is located at an asset position in the second diagonal line and 000 in which two $0's$ are located at asset positions in the second diagonal line, then it

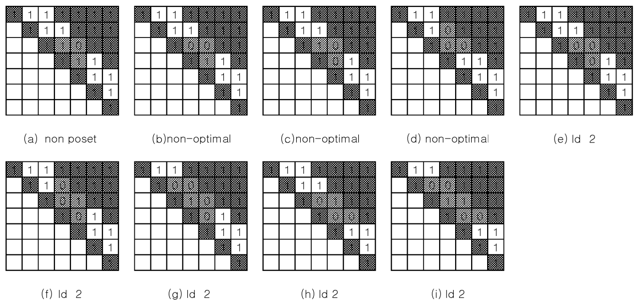

FIGURE 3. All possible cases to be considered for non-optimal labelled matrices.

is a non-optimal labelled matrix (see Figure 3(b), (c), and (d)). We want to count these non-optimal labelled matrices since they can be converted to matrices of LD 1 by relabelling operation as represented in Figure 4. If U_P^f has 000 in which one 0 is located at an asset position in the second diagonal line, then it is LD two(see Figure 3(e)). If U_P^f has 0010, 00100, 0100, and 001100 as (f), (g), (h), and (i) in Figure 3, then it is of LD two. Since the adjacency matrix of a comparability graph of LD two is the adjacency matrix of a graph of bandwidth two if we exchange 0 and 1 in the adjacency matrix of a comparability graph of LD two, the following lemma is logically equivalent to the Lemma 1 in [1]

Lemma 3.1. *Let A be a matrix which is not a non-poset matrix. A is a non-optimal labelled matrix of order n if and only if A has* 001, 100, *or* 000 (*with two* $0'$*s at asset positions in the second diagonal line consecutively as illustrated in Figure 3*(d)), *but none of* 000(*one* 0 *is located at an asset position in the second diagonal line*), 0010, 00100, 0100, *and* 001100 (*see Figure 3*(e), (f), (g), *and* (h), *and* (i)) *in the asset positions.*

This lemma gives the same consequences to Proposition 1.5. In Figure 3, (e) represents $\mathbf{1}+\mathbf{1}+\mathbf{1}$, ($f$), ($h$), and ($i$) represent $\mathbf{1}+\mathbf{3}$, respectively, and (g) represents $\mathbf{2}+\mathbf{2}$.

Throughout this section, let m_n denote $NOL(n)$ for convenience. Non-optimal labelled matrices of order n can be obtained by appending the n-th column and the n-th row to non-optimal labelled matrices of order $n-1$ at a time. Let $U_P^f = [a_{i,j}]_n$, $x = a_{n-2,n}$, $y = a_{n-1,n}$, $a = a_{n-3,n-1}$, $b = a_{n-2,n-1}$, $c = a_{n-4,n-2}$, and $d = a_{n-3,n-2}$. Let a_n, b_n, c_n, and d_n be the numbers of non-optimal labelled matrices of order n with $(x,y)=(1,1)$, $(x,y)=(1,0)$, $(x,y)=(0,1)$, and $(x,y)=(0,0)$, respectively. Clearly, we have

$$m_n = a_n + b_n + c_n + d_n \tag{7}$$

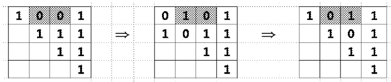

FIGURE 4. relabelling operation

Case 1 : Counting a_n.
It is clear that appending a column and a row to non-optimal labelled matrices of order $n-1$ with $(x, y) = (1, 1)$ cannot produce any non-optimal labelled matrices of order n. Therefore, we have $a_n = m_{n-1}$, and hence

$$a_n = a_{n-1} + b_{n-1} + c_{n-1} + d_{n-1} \tag{8}$$

Case 2: Counting b_n.
When $(x, y) = (1, 0)$, we have four subcases as follows:

Subcases $(a, b) = (1, 1)$, $(a, b) = (1, 0)$, and $(a, b) = (0, 1)$: There is no non-optimal labelled matrix of order n produced by this case, since we do not have 001, 100, or 000 at all(See Figure 3(b), (c), and (d)) with a_{n-1}, b_{n-1}, and c_{n-1}, respectively. They are counted by a_{n-1}, b_{n-1}, and c_{n-1}, respectively.

Subcase $(a, b) = (0, 0)$: In this case, relabeling(See Figure 4) for b results in $(x, y) = (0, 1)$, which cannot produce a matrix of LD one(See Figure 3(f)). This means that the relabeling fails. Therefore, this case does not give any contribution to b_n. In all, we have

$$b_n = a_{n-1} + b_{n-1} + c_{n-1} \tag{9}$$

Case 3 : Counting c_n.
When $(x, y) = (0, 1)$, we have four subcases as follows:

Subcase $(a, b) = (1, 1)$: This case produces a non-poset matrix, so it is out of the case. Therefore, this case does not contribute to c_n.

Subcase $(a, b) = (1, 0)$: Since $b = 0$, $x = 0$, and $y = 1$ cause a sequence of 001, this case may produce non-optimal labelled matrices of order n(see Figure 3(b)). If the non-optimal labelled matrices of order n just produced are purely new non-optimal labelled matrices, then the number of these matrices has nothing to do with m_{n-1}. That means there is no consecutive 0 in the matrix and no 0 in the asset position of the second diagonal line. It means the matrices we need to count are of LD one. But if the asset position d is 0 in the upper right matrices in Figure 5, then we will face the case in Figure 3(h). Hence we have $d = 1$. Therefore we have $n-4$ asset positions to be filled up by 0 or 1, hence the number of these matrices of LD one is 2^{n-4}. Now if the non-optimal labelled matrices of order n just produced are from the non-optimal labelled matrices of order $(n-1)$ or less, these non-optimal labelled matrices of order n should be converted to the

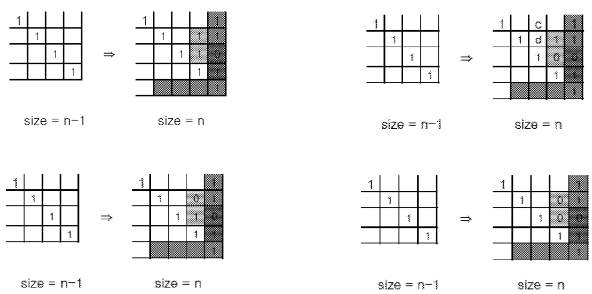

FIGURE 5. Subcases for type c_n

matrices of LD one. In order to do that, the asset positions c and d of upper right matrices in Figure 5 should be 1. Since $(x, y) = (1, 0)$, there was no new non-optimal labelled matrices of order $n-1$ produced in the previous step. Therefore, this case is counted by a_{n-2}.

Subcase $(a, b) = (0, 1)$: Since $b = 1$, $x = 0$, and $y = 1$, This case produces a non-poset matrix, so it is out of the case. Therefore, this case does not contribute to c_n.

Subcase $(a, b) = (0, 0)$: Since $a = 0$ and $b = 0$, it may produce non-optimal labelled matrices. And there are no new non-optimal labelled matrices in the current step(See Figure 3(d)). Therefore, it is counted by d_{n-1}.

In total, we have

$$c_n = a_{n-2} + d_{n-1} + 2^{n-4} \tag{10}$$

Case 4 : Counting d_n.
When $(x, y) = (0, 0)$, we have four subcases as follows:

Subcase $(a, b) = (1, 1)$. Since $x = 0$ and $y = 0$, this case may produce non-optimal labelled matrices of order n. If the non-optimal labelled matrices of order n just produced are purely new matrices, then the number of these matrices of LD one is 2^{n-3}. If the non-optimal labelled matrices of order n just produced are from the non-optimal labelled matrices of order $(n-1)$ or less, then this case is counted by a_{n-1} since $a = 1$ and $b = 1$. But if $c = 0$ and $d = 0$ of the first matrices in Figure 6, then a relabeling operation for $c = 0$ and $d = 0$ makes these matrices of LD three, since $x = 0$ makes $(n-3, n)$-entry 0. And these matrices cannot be converted to the matrices of LD one by any sequential relabeling operations since $b = 1$. Hence, these cases which are counted by d_{n-2} should be excluded. Therefore, this case is counted by $a_{n-a} - d_{n-2}$.

Subcase $(a, b) = (1, 0)$: Since $b = 0$, $x = 0$, and $y = 0$, it is clear that any relabeling operations fail to make the matrix for this case of LD one(See Figure 3(e)). Therefore, this case does not contribute to d_n.

Subcase $(a, b) = (0, 1)$: A relabeling operation for $x = 0$ and $y = 0$ makes these matrices of LD three, since $a = 0$ makes $(n-3, n)$-entry 0(See Figure 3, g). And these matrices cannot be converted to the matrices of LD one by any sequential relabeling operations since the only way to make $(n-3, n)$-entry 1 is to take a relabeling operation if it was $d = 0$. But if $d = 0$, this is

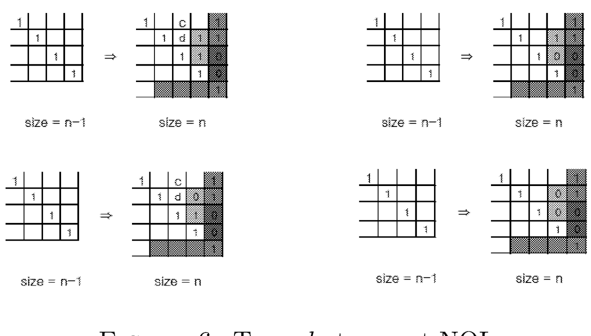

FIGURE 6. Type d_n to count NOL

the non-poset matrix, so it is impossible. Therefore, this case does not give any contribution to d_n.

Subcase $(a,b)=(0,0)$: Since $b=0$, $x=0$, and $y=0$, it is clear that any relabeling operations fail to make it of LD one(See Figure 3(e)). Therefore, this case does not contribute to d_n.

In all, we have

$$d_n = a_{n-1} - d_{n-2} + 2^{n-3}. \tag{11}$$

Let $M(x)=\sum m_n x^n$, $A(x)=\sum a_n x^n$, $B(x)=\sum b_n x^n$, $C(x)=\sum a_n x^n$, $D(x)=\sum d_n x^n$, and $K(x)=\sum 2^n x^n$. From (7), (8), (9), (10), and (11) with the generating functions above, we have

$$M(x) = A(x) + B(x) + C(x) + D(x), \tag{12}$$

$$(1-x)A(x) = xB(x) + xC(x) + xD(x), \tag{13}$$

$$(1-x)B(x) = xA(x) + xC(x), \tag{14}$$

$$C(x) = x^2 A(x) + xD(x) + (1/16)K(x), \tag{15}$$

and

$$(1+x^2)D(x) = xA(x) + (1/8)K(x), \tag{16}$$

Solve the Equations from (12) to (17) for $M(x)$, we have

$$(-16+32x+48x^3+16x^5)M(x) = (-3-x^2)K(x) \tag{17}$$

and the following recurrence relation:

Theorem 3.2. *The recurrence relation for $NOL(n)$ is*

$$m_n = 2m_{n-1} + 3m_{n-3} + m_{n-5} + 13\cdot 2^{n-6}, \quad n\geq 6, \tag{18}$$

where $m_1=0$, $m_2=0$, $m_3=2$, $m_4=7$, $m_5=21$.

By adding several terms obtained from the boundary conditions of the Equation (18), we have

$$\begin{aligned} M(x) &= -\frac{x^3(-2+x-x^2+x^3)}{(-1+2x)(-1+2x+3x^3+x^5)} \\ &= -\frac{1}{2} + \frac{1/2-2x+2x^2+x^3/2+2x^4+x^5/2}{-16+32x+48x^3+16x^5} \end{aligned} \tag{19}$$

where $m_1=0$, $m_2=0$, $m_3=2$, $m_4=7$, $m_5=21$.

The partial fraction method is used to find approximation for the number m_n. From Equation (19), we have

(20)

$$\begin{aligned} M(x) = &-\frac{1}{2} + \frac{0.150709...}{0.399372...-x} + \frac{0.5}{2x-1} \\ &+ \frac{0.930105...+0.001194...x}{2.27055...-0.304016...x+x^2} + \frac{0.234888...+0.149515...x}{1.10279...+0.703387...x+x^2}. \end{aligned}$$

The coefficients of x^n in $\frac{0.930105...+0.001194...x}{2.27055...-0.304016...x+x^2}$ and $\frac{0.234888...+0.149515...x}{1.10279...+0.703387...x+x^2}$ tend to zero as n tends to ∞. The coefficient of x^n in $\frac{0.5}{2x-1}$ is

$$-2^{n-1}$$

and coefficient of x^n in $\frac{0.150709...}{0.399372...-x}$ is $0.150709 \cdot 0.399372^{-1-n}$. Hence we have the following result.

Theorem 3.3. *The asymptotic number of non-poset matrices is*

$$m_n \sim C_2 \cdot r_2^{-1-n} - 2^{n-1} \tag{21}$$

where $C_2 = 0.150709...$ *and* $r_2 = 0.399372...$

From Theorem 1.6, 2.1, and 3.2, the number of posets of LD two with optimal labellings is found as follows.

Theorem 3.4. *Let* $LD_i(n)$ *be the number of optimally labelled posets of linear discrepancy* $i = 1, 2$, *respectively. Then we have*

$$\begin{aligned} LD_2(n) =& 2^{2n-3} - LD_1(n) - NP(n) - NOL(n) - 1 \\ =& 2^{2n-3} - 2^{n-1} \\ & - 3\sum_{i=0}^{n-4} 2^{2i}\left(-\frac{1}{4} + \frac{D}{r_1^{n-4-i}} + \frac{E}{r_2^{n-4-i}}\right) + \frac{D}{r_1^{n-3}} + \frac{E}{r_2^{n-3}} - \frac{1}{4} \\ & - NOL(n) \end{aligned} \tag{22}$$

where $n \geq 5$, $LD_2(3) = 1$, $LD_2(4) = 7$, $D = \frac{5}{8} + \frac{19}{8\sqrt{17}}$, $E = \frac{5}{8} - \frac{19}{8\sqrt{17}}$, $r_1 = \frac{-3+\sqrt{17}}{4}$, *and* $r_2 = \frac{-3-\sqrt{17}}{4}$.

From Theorem 2.3, and 3.3 the asymptotic number of posets of LD two with optimal labelling is found as follows.

Theorem 3.5. *Let* $LD_i(n)$ *be the number of optimally labelled posets of linear discrepancy* $i = 1, 2$, *respectively. Then we have*

$$\begin{aligned} LD_2(n) \sim& \ 2^{2n-3} - LD_1(n) - NP(n) - NOL(n) - 1 \\ =& \ C_1 \cdot r_1^{-1-n} - C_2 \cdot r_2^{-1-n} \end{aligned} \tag{23}$$

where $C_1 = 0.0436093...$, $r_1 = 0.280776...$, $C_2 = 0.150709...$, *and* $r_2 = 0.399372....$

Note that the labelled matrices of LD two converted to the adjacency matrices of bandwidth two by exchanging 0 and 1 in the labelled matrices. And the non-optimal labelled matrices converted to the non-optimal adjacency matrices of graphs which are not labelled optimally, which looks like bandwidth two, but they could be relabeled to be of bandwidth one(See [1]) by exchanging 0 and 1 in the labelled matrices. Moreover the non-poset matrices are divided into two sets when the non-poset matrices are converted to complementary matrices by exchanging 0 and 1 in the labelled matrices. One of them is the set of matrices which becomes the adjacent matrices of graphs of bandwidth 2, say $NIB_2(n)$, and the other is a set of matrices which becomes the non-optmal adjacency matrices of graphs, let us call it $NINO(n)$. Therfore, we have the following corollaries.

Corollary 3.6. *The number of graphs of order n and of bandwidth two is the number of optimally labelled matrices of LD two plus the number of* $NIB_2(n)$.

Corollary 3.7. *The number of non-optimal matrices of graphs of order n is the number of the non-optimal labelled matrices plus the number of $NINO(n)$.*

n	$LD_1(n)$	NOL	NP	$LD_2(n)$	NIB_2(n)	NINO(n)	Bw_2(n)
3	3	2	1	1	0	1	1
4	7	7	7	10	0	7	10
5	15	21	39	52	11	28	63
6	31	61	195	224	94	101	318
7	63	169	919	896	572	347	1468
8	127	455	4171	3438	3024	1147	6462
9	255	1204	18447	12861	14747	3700	27608
10	511	3144	80067	47349	68329	11738	115678
11	1023	8130	342631	172503	305861	36770	478364
12	2047	20873	1450171	624060	1336092	114079	1960152
13	4095	53297	6084351	2246864	5733044	351307	7979908
14	8191	135516	25347699	8063025	24272167	1075532	32335192
15	16383	343451	104989015	28868878	101711598	3277417	130580476
16	32767	868235	432771307	103198602	422821614	9949693	526020216
17	65535	2190515	1776727407	368500190	1746613522	30113885	2115113712
18	131071	5517928	7269466659	1314818933	7178549251	90917408	8493368184
19	262143	13882573	29656822087	4688771564	29382888660	273933427	34071660224
20	524287	34893134	120689268763	16714267287	119865287777	823980986	136579555064
21	1048575	87634271	490100927199	59566203842	487625841710	2475085489	547192045552
22	2097151	219958744	1986559226067	212241973589	1979133015435	7426210632	2191374989024
23	4194303	551818754	8039391160375	756145848775	8017130766793	22260393582	8773276615568
24	8388607	1383830766	32489338444363	2693641425095	32422664467289	66673977074	35116305892384
25	16777215	3469246642	131138983698255	9595018633215	130939413773297	199569924958	140534432406512
26	33554431	8695215305	528764372161155	34176852490420	528167341824300	597030336855	562344194314720

TABLE 1

The corresponding numbers are given in Table 1.

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GAB-BYUNG CHAE, DIVISION OF MATHEMATICS AND INFORMATIONAL STATISTICS, WONKWANG UNIVERSITY, IKSAN, JEOLLABUKDO, KOREA
E-mail address: rivendell@wonkwang.ac.kr